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1979 J. Phys. A: Math. Gen. 12 1425

(http://iopscience.iop.org/0305-4470/12/9/011)

J. Phys. A: Math. Gen., Vol. 12, No. 9, 1979. Printed in Great Britain

Transformations between inertial and rotating frames of reference

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Received 26 January 1978, in final form 29 November 1978

Abstract. The problem of relating measurements made in an inertial (laboratory) frame to measurements made in a rotating frame is attacked through the derivation from first principles of the appropriate transformation equations. These are then used to derive a metric for a rotating system in which the energy tensor is everywhere zero, i.e. a rotating massless system. This metric is used to obtain descriptions of a variety of phenomena in rotating systems. Similarities between results produced by the metric approach and results produced by the technique of associating linearly-moving Lorentz frames instantaneously with points within the rotating system are indicated. Emphasis is laid throughout upon the way in which measurements made by different techniques are interpreted and related.

1. Introduction

The rotating coordinate system presents a problem of particular interest, as is witnessed by the large number of publications on the subject which have appeared since the inception of relativity. However, although numerous attempts have been made to find a suitable coordinate transformation from an inertial frame to a rotating frame, or to construct a metric for a rotating frame, none of them has yet been verified by experiment. This paper is concerned with the derivation of a coordinate transformation and metric for a rotating system and, as will be shown, produces results consistent with all experiments to date on rotating systems. Particular emphasis is laid on the interpretation of measurements made in rotating systems.

The metric is derived from the Minkowski space–time of a 'laboratory frame' by first deriving the general transformations to the rotating system. The derivation of the transformations relies on the assumption of radar measurement as a fundamental and valid method of measuring distance in any space time and the transformations are applicable to rotating systems in which the energy tensor is everywhere zero, i.e. a rotating massless system.

It is important to realise that since the formulation of a 'general metric' for a rotating system depends on the coordinate system used, the metric will be a function of the way in which measurements are made within the rotating system. For example, if A and B are two points in synchronous rotation with the system then an observer at point A who measures the distance to point B by a radar technique will, in general, obtain a different result by walking along the shortest path from A and B and noting the distance that he has travelled. Obviously these two forms of measurement can be related but the metric itself will be a function of which measurement procedure is chosen to describe

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coordinate distance. The second of the measurement procedures described above we shall call 'collated proper measurement' since it involves the successive measurement by an observer in his own local frame of a series of infinitesimally small steps between A and B. This technique is equivalent to considering events as they are described by an infinite number of observers, stationary with respect to the rotating system, who collate their observations at a single data centre.

Why should we consider these different methods of measurement? It would perhaps seem more logical to confine our attention to one technique, say radar measurement, and then to define all our measurements in terms of this technique. However, in practice this is not sufficient for one generally makes many different types of measurement of the same event and it is important to consider as many different forms of measurement of the same event as possible in order to learn about the nature of the space. Also, on a practical note, an observer travelling from A to B will be interested in the shortest distance that he need travel in going from A to B as well as the radar distance of B from A.

In the paper we consider the derivation of the metric and then examine applications of the metric to descriptions of various phenomena as seen in a rotating system. The metric produces results consistent with previous work and where this occurs the correlation has been pointed out. We have not found any instances where the metric produces results inconsistent with any experimental work on rotating systems.

However, it is important to emphasise in this introduction that the results obtained are a function of the way in which measurements are made and interpreted in rotating systems.

2. The derivation of the coordinate transformation to a rotating system

Consider figure 1 which shows how a radar measurement, made by an observer at A, of the distance AB between two points A and B which are fixed in a rotating system, appears when drawn in the inertial frame, S, of the laboratory. The system is rotating with respect to the laboratory at angular velocity ω about the centre of rotation, O. In the following analysis unprimed symbols refer to the inertial frame S and primed symbols refer to measurements made by the rotating observer at A. A is situated at a *constant* inertial distance R from the centre of rotation O. $d\theta_i$ is the instantaneous value of $d\theta$ such that $d\theta = d\theta_i + \omega dt$.

A radar signal is sent out by A when A is at A_1 and B at B_1. This signal is received by B when B is at B_2 and A at A_2. B instantaneously retransmits the signal back to A and A receives it when he is at A_3 and B is at B_3. If the time taken for the signal to travel from A_1 to B_2 is dt_1 and the time taken for the signal to travel from B_2 to A_3 is dt_2 then

$$d\sigma_1 = c\ dt_1 \qquad d\sigma_2 = c\ dt_2 \tag{1), (2}$$

$$d\phi_1 = \omega\ dt_1 \qquad d\phi_2 = \omega\ dt_2. \tag{3), (4}$$

From figure 1 and using equations (3) and (4) we see that

$$d\sigma_1^2 = dR^2 + R^2(d\theta_i + d\phi_1)^2 = dR^2 + R^2(d\theta_i + \omega\ dt_1)^2 \tag{5}$$

and

$$d\sigma_2^2 = dR^2 + R^2(d\theta_i - d\phi_2)^2 = dR^2 + R^2(d\theta_i - \omega\ dt_2)^2 \tag{6}$$

in which R is constant and dR is an incremental distance in the inertial radial direction.

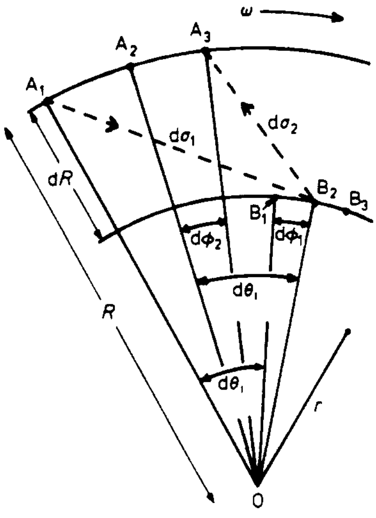

Figure 1. The appearance of a radar measurement, made by A, of the distance between points A and B on a rotating system, drawn in the inertial frame, S, of the laboratory in which the system is rotating at angular velocity ω about the centre of rotation O.

Using equations (1) and (2) to substitute into equations (5) and (6) for dt_1 and dt_2 we find after solving the resulting quadratic equations for $d\sigma_1$ and $d\sigma_2$ that (Arzeliès 1966, Ashworth and Jennison 1976)

$$\tfrac{1}{2}(d\sigma_1 + d\sigma_2) = [dR^2/(1 - R^2\omega^2/c^2) + R^2\, d\theta_i^2/(1 - R^2\omega^2/c^2)^2]^{1/2} = \tfrac{1}{2}c(dt_1 + dt_2). \tag{7}$$

Following Arzeliès (1966) by letting $\frac{1}{2}(dt_1 + dt_2) = dt$ means that the 'radar' distance $d\sigma$ between A and B is given by $d\sigma = c\,dt$ according to an inertial observer in the laboratory frame. Hence

$$d\sigma^2 = \frac{dR^2}{(1 - R^2\omega^2/c^2)} + \frac{R^2\, d\theta_i^2}{(1 - R^2\omega^2/c^2)^2} = c^2\, dt^2. \tag{8}$$

However, the rotating observer A will measure all distances within his own infinitesimally small local frame in terms of Euclidean geometry, and this geometry may be extended to a Euclidean plane covering the whole rotating system (Arzeliès 1966) as shown in figure 2. Therefore according to the rotating observer A, the radar distance to B is given by $d\sigma'$ where, according to special relativity

$$d\sigma'^2 = dR^2 + R^2\, d\theta'^2 = c^2\, dt'^2 \tag{9}$$

since both R and dR are normal to the instantaneous velocity vector at A. Equations (8) and (9) can be written in the form

$$0 = dR^2 + \frac{R^2\, d\theta_i^2}{(1 - R^2\omega^2/c^2)} - c^2\left(1 - \frac{R^2\omega^2}{c^2}\right) dt^2 = dR^2 + R^2\, d\theta'^2 - c^2\, dt'^2 \tag{10}$$

which can be satisfied by requiring that

$$d\theta' = \frac{d\theta_i}{(1 - R^2\omega^2/c^2)^{1/2}} = \frac{d\theta - \omega\, dt}{(1 - R^2\omega^2/c^2)^{1/2}} \tag{11}$$

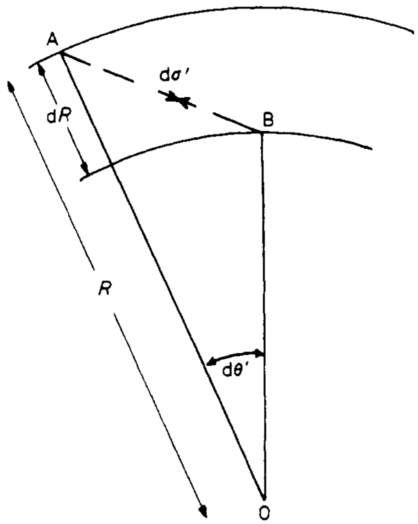

Figure 2. To show how the Euclidean geometry of the infinitesimally small local frame of the rotating observer A may be extended to a Euclidean plane covering the whole rotating system.

and

$$dt' = dt(1 - R^2\omega^2/c^2)^{1/2}. \tag{12}$$

Equations (11) and (12) reduce to $d\theta' = d\theta - \omega\, dt$ and $dt' = dt$ when $R = 0$ which is the usual Galilean transformation assumed for a stationary observer at the centre of a rotating coordinate system.

The 'radial' transformation may be obtained by sending a radar signal from the observer A to the centre of rotation of the system where it is reflected back and received by A after a time $2t'_{AO}$ has elapsed. Since $2t'_{AO} = 2t_{AO}(1 - R^2\omega^2/c^2)^{1/2}$ by equation (12), and since $R = ct_{AO}$, we have

$$2t'_{AO} = (2R/c)(1 - R^2\omega^2/c^2)^{1/2}. \tag{13}$$

The observer at A will define the radial distance to the centre as being given by $R' = ct'_{AO}$ since local measurements of the velocity of light will give a value c. Provided that we remember that radial measurements have been made by radar techniques then we can set (Jennison 1964)

$$R' = R(1 - R^2\omega^2/c^2)^{1/2} \tag{14}$$

in which both R and R' are constants since the observer is fixed at A.

Let us assume that the observer at A now wishes to find the distance of some arbitrary point P from the centre O. An observer at O will measure, by radar, the distance to P as $r = ct_{PO}$ but A knows that $t'_{PO} = t_{PO}(1 - R^2\omega^2/c^2)^{1/2}$ and so the radar distance of P from O written in the coordinates of the observer at A will be $r' = ct'_{PO}$, whence

$$r' = r(1 - R^2\omega^2/c^2)^{1/2} \tag{15}$$

which is valid for all r and gives equation (14) when $r = R$ and $r' = R'$. Hence, all radial coordinate distances, according to the rotating observer A, are reduced by the scaling factor $(1-R^2\omega^2/c^2)^{1/2}$. By analogy with proofs of the Lorentz transformations (e.g. Atwater 1974) we shall assume that distances measured transverse to the direction of relative velocity and acceleration, i.e. in the z, z' direction, must be unaffected by the motion. Hence

$$z' = z. \tag{16}$$

From equations (11) and (12) it is possible to derive a further equation which relates the angular velocity measured by an observer at rest in the rotating system to that measured by an observer at rest in the laboratory frame. If we fix $\theta = \text{constant}$ then

$$\frac{d\theta'}{dt'} = \frac{-\omega}{(1-R^2\omega^2/c^2)} = -\omega' \tag{17}$$

where $-\omega'$ is the angular velocity of the laboratory frame as measured in the coordinates of the system S' where S' is the rotating frame according to the observer at A. Hence ω' is the angular velocity of rotation of the system S as measured by an observer who is stationary in S'. This expression for ω' is identical to the one derived by Irvine (1964) and would appear at first sight to be incompatible with the equation given by Jennison (1964) which is of the form

$$\Omega' = \frac{\omega}{(1-R^2\omega^2/c^2)^{1/2}}. \tag{18}$$

However, the angular velocity Ω' given by equation (18) refers to the rate at which the universe appears to rotate with respect to an observer who measures the rotation of the universe by measuring the time interval for one revolution in a particular manner: he measures the time interval between successive transits of some point over which he passes. An assumption implicit in this kind of measurement of angular velocity is that locally there are 2π radians in one revolution, both for the rotating observer and for the observer fixed in the laboratory frame. Hence equation (18) assumes $\theta' = \theta - \omega t$ and this assumption accounts for the difference of $(1-R^2\omega^2/c^2)^{1/2}$ between equations (17) and (18).

Using equations (15) and (17) we can easily show that

$$(1-R^2\omega^2/c^2) = (1+R'^2\omega'^2/c^2)^{-1} \tag{19}$$

which, together with equations (11), (12), (15) and (16), enables the coordinate transformations between an inertial frame of reference $S(r, \theta, z, t)$ and a rotating frame of reference $S'(r', \theta', z', t')$ to be written as

$$\begin{aligned}
r' &= r(1-R^2\omega^2/c^2)^{1/2} & r &= r'(1+R'^2\omega'^2/c^2)^{1/2} \\
\theta' &= \frac{(\theta-\omega t)}{(1-R^2\omega^2/c^2)^{1/2}} & \theta &= \frac{(\theta'+\omega' t')}{(1+R'^2\omega'^2/c^2)^{1/2}} \\
z' &= z & z &= z' \\
t' &= t(1-R^2\omega^2/c^2)^{1/2} & t &= t'(1+R'^2\omega'^2/c^2)^{1/2}
\end{aligned} \tag{20}$$

in which R, ω, R' and ω' are constants.

The symmetry of these equations and their similarity to the Lorentz transformations is strikingly apparent. Equations (20) reduce to the Galilean rotational transformations for $R = 0$, as indeed they must. It should be borne in mind in what follows that the above transformations apply to transformations between inertial observers in S and observers in S' who are rotating at *constant* inertial radial distance R from the centre of rotation O.

3. The metric of the rotating system

In this section we consider the metric obtained when the transformations given in equation (20) are applied to the flat space–time of the Minkowski metric. *The resulting metric corresponds to a description of the rotating system as made by an observer at a constant inertial radial distance R from the centre of rotation* O *of the rotating system and who is in synchronous rotation with the system.* The coordinate system in which this observer makes his measurements has its origin at the centre of rotation of the system. The cylindrical form of the Minkowski metric in the laboratory system $S(r, \theta, z, t)$ is given by

$$\mathrm{d}s^2 = -\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - \mathrm{d}z^2 + c^2\,\mathrm{d}t^2 \tag{21}$$

which transforms by equations (20) to give

$$\begin{aligned}\mathrm{d}s^2 = &-(1+R'^2\omega'^2/c^2)\,\mathrm{d}r'^2 - r'^2\,\mathrm{d}\theta'^2 - 2\omega' r'^2\,\mathrm{d}\theta'\,\mathrm{d}t' - \mathrm{d}z'^2\\ &+ c^2[1+\omega'^2(R'^2 - r'^2)/c^2]\,\mathrm{d}t'^2\end{aligned} \tag{22}$$

for the metric in $S'(r', \theta, z', t')$. It is important to note here that both R and R' are constants for an observer at a fixed point within the rotating system. The metric of equation (22), therefore, gives a description of the rotating system for the observer situated at the said fixed point. If the observer moves then R and R' change and the observer then effectively has a new metric to apply.

The spatial line elements of the rotating system can be obtained from equation (22) by using the definition (e.g. Atwater 1974)

$$\mathrm{d}\sigma^2 = -(g_{\mu\nu} - g_{\mu 0}g_{\nu 0}/g_{00})\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{23}$$

from which, since the suffix 0 refers to the time coordinate,

$$\mathrm{d}\sigma'^2 = \mathrm{d}r'^2(1+R'^2\omega'^2/c^2) + r'^2(1+\omega'^2R'^2/c^2)[1+\omega'^2(R'^2-r'^2)/c^2]^{-1}\,\mathrm{d}\theta'^2 + \mathrm{d}z'^2 \tag{24}$$

or, alternatively,

$$\mathrm{d}\sigma'^2 = \mathrm{d}r^2 + r^2(1-R^2\omega^2/c^2)(1-r^2\omega^2/c^2)^{-1}\,\mathrm{d}\theta'^2 + \mathrm{d}z^2. \tag{25}$$

Since the geometry *within* the infinitesimally small local frame, S', is Euclidean a rotating observer in S' who holds a torch and uses it to emit a flash of light would 'see' the light recede from him as a spherical wavefront travelling with velocity c. This phenomenon is described mathematically by using equation (22). For a light signal, $\mathrm{d}s = 0$, which enables equation (22) to be solved for $\mathrm{d}t'$ giving

$$\mathrm{d}t' = \frac{\omega r^2(1-R^2\omega^2/c^2)\,\mathrm{d}\theta'}{c^2(1-r^2\omega^2/c^2)} \pm \frac{(1-R^2\omega^2/c^2)^{1/2}}{c(1-r^2\omega^2/c^2)^{1/2}}\left(\mathrm{d}r^2 + \frac{(1-R^2\omega^2/c^2)r^2\,\mathrm{d}\theta'^2}{(1-r^2\omega^2/c^2)} + \mathrm{d}z^2\right)^{1/2} \tag{26}$$

which, by equation (25) can be written in the form

$$\mathrm{d}t' = \frac{\omega r^2(1-R^2\omega^2/c^2)\,\mathrm{d}\theta'}{c^2(1-r^2\omega^2/c^2)} \pm \frac{(1-R^2\omega^2/c^2)^{1/2}\,\mathrm{d}\sigma'}{c(1-r^2\omega^2/c^2)^{1/2}}$$

or, by using the transformation equations,

$$d\sigma' = \frac{c(dt - \omega r^2\, d\theta/c^2)}{(1 - r^2\omega^2/c^2)^{1/2}}. \tag{27}$$

But in the proper frame of the observer $r = R$ and $d\sigma' = c\,dT'$, where dT' is the time taken by a ray of light in travelling a distance $d\sigma'$ according to a rotating observer who makes actual '*one-way*' measurements in S'. Hence

$$dT' = \frac{dt - \omega R^2\, d\theta/c^2}{(1 - R^2\omega^2/c^2)^{1/2}} \tag{28}$$

which is the same time transformation as that obtained by applying the instantaneous Lorentz-frame approach to measurements in rotating systems, that is according to such Lorentz-transformation time-transforms as

$$dT' = \frac{dt - v\,dx/c^2}{(1 - v^2/c^2)^{1/2}}. \tag{29}$$

Replacing v by $R\omega$ and dx by $R\,d\theta$ in equation (29) gives equation (28). Alternatively equation (28) may be written in the form

$$dT' = dt' - R'^2\omega'\, d\theta'/c^2. \tag{30}$$

We now look at some applications of the theory so far derived.

3.1. The velocity of a particle in the rotating system

The velocity u' of a particle, as measured locally (at $r = R$) by an observer in synchronous rotation with the system, is given by

$$u' = (d\sigma'/dT')_{r=R} \tag{31}$$

and hence, by equations (25) and (28)

$$u' = c[(c^2 - R\omega u_\theta)^2 - (c^2 - u^2)(c^2 - R^2\omega^2)]^{1/2}(c^2 - \omega R u_\theta)^{-1} \tag{32}$$

in which,

$$\left.\begin{aligned} &u_r = dr/dt,\ u_\theta = r\, d\theta/dt,\ u_z = dz/dt, \\ &u^2 = u_r^2 + u_\theta^2 + u_z^2 \end{aligned}\right\}\ \text{at } r = R. \tag{33}$$

Letting

$$u'^2 = u_r'^2 + u_\theta'^2 + u_z'^2 \tag{34}$$

gives

$$u_r' = \left(\frac{dr}{dT'}\right)_{r=R} = \frac{u_r(1 - R^2\omega^2/c^2)^{1/2}}{1 - \omega R u_\theta/c^2} \tag{35}$$

$$u_\theta' = \left(r\frac{d\theta'}{dT'}\right)_{r=R} = \frac{u_\theta - \omega R}{1 - \omega R u_\theta/c^2} \tag{36}$$

and

$$u_z' = \left(\frac{dz}{dT'}\right)_{r=R} = \frac{u_z(1 - R^2\omega^2/c^2)^{1/2}}{1 - \omega R u_\theta/c^2} \tag{37}$$

where equations (35)–(37) are identical to the velocity addition formulae of special relativity.

However, the apparent or coordinate velocity of a particle, $u'_{\rm app}$, according to a rotating observer is given by

$$u'_{\rm app} = \mathrm{d}\sigma'/\mathrm{d}t' \tag{38}$$

from which, by means of equations (20) and (25), we see that

$$u'_{\rm app} = \frac{c[(c^2 - r\omega u_\theta)^2 - (c^2 - u^2)(c^2 - r^2\omega^2)]^{1/2}}{(c^2 - r^2\omega^2)^{1/2}(c^2 - R^2\omega^2)^{1/2}}, \tag{39}$$

in which $u_\theta = r\ \mathrm{d}\theta/\mathrm{d}t$. It is important to realise that $u'_{\rm app}$ as given by equation (39) is not a velocity *measured* within the rotating system.

The components of the apparent velocity are

$$u'_{r\,\rm app} = u_r(1 - R^2\omega^2/c^2)^{-1/2}, \tag{40}$$

$$u'_{\theta\,\rm app} = (u_\theta - \omega r)(1 - r^2\omega^2/c^2)^{-1/2}(1 - R^2\omega^2/c^2)^{-1/2}, \tag{41}$$

and

$$u'_{z\,\rm app} = u_z(1 - R^2\omega^2/c^2)^{-1/2}. \tag{42}$$

A free particle travelling with constant velocity u in the r, θ plane of the laboratory system S will have components of velocity in the system S given by

$$u_r = u(r^2 - a^2)^{1/2}/r, \qquad u_\theta = ua/r, \qquad u_z = 0 \tag{43}$$

where a is the radial distance to the point of closest approach of the particle to the axis of rotation of the system.

3.2. The velocity of light in the rotating system

From equations (39) and (43) we can obtain the apparent velocity of light $c'_{\rm app} = \mathrm{d}\sigma'/\mathrm{d}t'$ in the rotating system by setting $u = c$, thus obtaining

$$c'_{\rm app} = c^2(c \pm a\omega)(c^2 - r^2\omega^2)^{-1/2}(c^2 - R^2\omega^2)^{-1/2}. \tag{44}$$

Equation (44) does not give the velocity of light as it would actually be measured within the rotating system, as 'proper' measurements of velocity are made either by 'there and back' radar measurements or by means of two observers who, by using synchronised clocks, note the time when a particle passes through their own localities. The 'proper distance' between the two observers divided by this 'proper time interval' will give the 'proper velocity'. Using these methods the velocity of light in the immediate vicinity of any observer, whether or not he is accelerated, will always be c. This can be shown to be true in the case of the rotating observer by setting $u = c$ into equation (32), whereupon

$$c' = c. \tag{45}$$

3.3. Equations of motion

The equations of motion of free mass points under the influence of gravitational and inertial 'forces' are described by the equation (e.g. Atwater 1974 p 120)

$$\ddot{x}^\mu + \left\{ {\mu \atop \nu\ \ \lambda} \right\} \dot{x}^\nu \dot{x}^\lambda = 0 \tag{46}$$

where the 'dot' denotes differentiation with respect to s and where $\{_{\nu}{}^{\mu}{}_{\lambda}\}$ denote Christoffel symbols of the second kind. Equation (46) gives the geodesic equations whose character is determined by the metric tensor of the coordinate manifold in which the point mass moves. Electromagnetic forces may be included into the equations of motion in a covariant way by adding an electromagnetic force term f^{μ} to the geodesic equation, thus giving

$$\ddot{x}^{\mu}+\left\{\begin{matrix} & \mu & \\ \nu & & \lambda\end{matrix}\right\}\dot{x}^{\nu}\dot{x}^{\lambda}=f^{\mu}. \tag{47}$$

We shall now use equation (46) together with the metric of equation (22) to derive the equations of geodesics within the rotating system according to a single observer who is stationary within the system and is situated at inertial distance R from the axis of rotation O. Geodesics fall into three classes: geodesics (the paths of free particles) within the rotating system, null geodesics (light paths) within the rotating system and spatial geodesics (the shortest spatial distance between two points) within the rotating system. We shall confine our attention to the plane $z=z'=0$.

3.4. Geodesics

The geodesic equations are obtained from equation (46) by using the metric tensor derived from equation (22) in the usual manner and are found to be, in differential form,

$$\frac{\mathrm{d}\theta'}{\mathrm{d}r}=\pm\frac{(au\pm\omega r^2)}{ur(r^2-a^2)^{1/2}(1-R^2\omega^2/c^2)^{1/2}} \tag{48}$$

where $r=a$, $r'=a'$ is the point of closest approach of the curve to the centre of the coordinate system and u is the velocity of a free particle travelling in the laboratory frame $S(r,\theta,z,t)$. Since u is independent of r, equation (48) may be integrated to give the equation for a geodesic:

$$\theta'=\pm(1-R^2\omega^2/c^2)^{-1/2}[\cos^{-1}(a/r)\pm(r^2-a^2)^{1/2}\omega/u] \tag{49}$$

if $\theta'=0$ when $r=a$. Also, using equations (25) and (48) to evaluate $\mathrm{d}\sigma'$ we find that

$$\mathrm{d}\sigma'=\frac{r[(c^2\pm\omega au)^2-(c^2-u^2)(c^2-\omega^2r^2)]^{1/2}\,\mathrm{d}r}{u(r^2-a^2)^{1/2}(c^2-\omega^2r^2)^{1/2}} \tag{50}$$

which, since u is independent of r, can be integrated to give

$$\sigma'=\frac{(c^2\pm a\omega u)}{\omega u}E(x,k)-\frac{(c^2-u^2)(r^2-a^2)^{1/2}(c^2-\omega^2r^2)^{1/2}}{u[(c^2-a\omega u)^2-(c^2-u^2)(c^2-\omega^2r^2)]^{1/2}} \tag{51}$$

where $E(x,k)$ is an elliptic integral of the second kind,

$$x=\sin^{-1}\left(\frac{(c^2\pm a\omega u)\omega(r^2-a^2)^{1/2}}{(c^2-\omega^2a^2)^{1/2}[(c^2\pm a\omega u)^2-(c^2-u^2)(c^2-\omega^2r^2)]^{1/2}}\right) \tag{52}$$

and

$$k=(c^2-u^2)^{1/2}(c^2-\omega^2a^2)^{1/2}(c^2\pm a\omega u)^{-1}. \tag{53}$$

σ', as derived above, is the length of a geodesic in the system S'. It is also readily shown that

$$dt' = \pm\frac{r(1-R^2\omega^2/c^2)^{1/2}\,dr}{u(r^2-a^2)^{1/2}} \tag{54}$$

which, since u is independent of r, integrates to give

$$t' = (r^2-a^2)^{1/2}(1-R^2\omega^2/c^2)^{1/2}u^{-1}. \tag{55}$$

3.5. *Null geodesics*

The equations of null geodesics may be obtained by setting $u = c$ in equations (49), (51) and (55). Equation (49) becomes

$$\theta' = \pm(1-R^2\omega^2/c^2)^{-1/2}[\cos^{-1}(a/r)\pm(r^2-a^2)^{1/2}\omega/c], \tag{56}$$

where $\theta' = 0$ when $r = a$ and equation (51) gives

$$\sigma' = \frac{1}{\omega}(c \pm a\omega)\sin^{-1}\left\{\frac{(\omega/c)(r^2-a^2)^{1/2}}{(1-\omega^2a^2/c^2)^{1/2}}\right\} \tag{57}$$

for the length of a ray path in the system S'. Also, equation (55) gives

$$t' = \frac{(r^2-a^2)^{1/2}}{c}\left(1-\frac{R^2\omega^2}{c^2}\right)^{1/2}. \tag{58}$$

3.6. *Spatial geodesics*

If we define a spatial geodesic as the shortest distance between any two points in space rather than in space–time then a spatial geodesic in a given coordinate system will be the path followed by a particle travelling at an infinite velocity as measured in that coordinate system (Ashworth and Davies 1977). Setting $u' = \infty$ in Eq. (32) gives

$$u_\theta = c^2/\omega R \tag{59}$$

But, $u_\theta = ua/R$ by equation (43). Therefore

$$u = c^2/a\omega. \tag{60}$$

A particle travelling with an infinite velocity in S' would therefore have a velocity of $c^2/a\omega$ in S. As both u and u' are greater than the velocity of light, c, for a particle travelling along a spatial geodesic, it is evident that no real free particle can ever travel along a spatial geodesic in the frame S'. To find the equations of a spatial geodesic we let $u = c^2/a\omega$ in equations (49) and (51), thus obtaining

$$\theta' = \pm(1-\omega^2R^2/c^2)^{-1/2}[\cos^{-1}(a/r)-(r^2-a^2)^{1/2}a\omega^2/c^2] \tag{61}$$

and

$$\sigma' = (1-a^2\omega^2/c^2)^{1/2}(r^2-a^2)^{1/2}. \tag{62}$$

For $R = 0$, equations (49) and (61) are identical to equations given by Arzeliès (1966).

Setting $a = 0$ in equation (62) gives $\sigma' = r$. Therefore the rotating observer situated at inertial radial distance R from the centre of rotation infers that the shortest 'walking' distance between himself and the centre of rotation is R provided that during the 'walk' his velocity relative to the rotating system is infinitesimally small. This would at first sight appear to conflict with equation (14) which states that this same observer will measure the radar distance to the centre of rotation to be $R(1-R^2\omega^2/c^2)^{1/2}$. However, the conflict is resolved when one begins to interpret the two types of measurement. The distance $R'=R(1-R^2\omega^2/c^2)^{1/2}$ is the shortest distance to the centre according to *a single rotating observer who always remains at the same radial distance from the centre* and who, therefore, is always travelling at the same linear speed, $R\omega$, with respect to an inertial observer at the centre. The distance $\sigma'_{RO}=R$ is this same observer's *interpretation* of the shortest distance to the centre that would be measured by an observer who actually crossed the system taking measurements during the journey but who was stationary with respect to the rotating system as each successive measurement was made. In deriving equation (62) all parts of the rotating system were considered to be in synchronous rotation, i.e. the observer crossing the system was assumed to have a linear speed (at the instant he took a measurement) with respect to an inertial observer at the centre of rotation, given by $r\omega$ where r varies from point to point as the observer crosses the rotating system and ω is constant and independent of r. The fixed observer at R is therefore interpreting the measurements made by this imaginary observer who takes an infinite number of measurements as he crosses the rotating system. Each time a measurement is made by the imaginary observer he is assumed to be in synchronous rotation with the system and therefore at each successive measurement he has a different linear speed $r\omega$. When the fixed rotating observer at $r=R$ interprets this infinite number of measurements he takes into account the fact that each measurement was made whilst the imaginary observer was travelling at a different velocity, thus coming to the conclusion that $\sigma'_{RO}=R$.

On the other hand, the distance $R'=R(1-R^2\omega^2/c^2)^{1/2}$ does not demand that ω should be independent of r as it is a measurement made by a single, fixed observer rotating at constant speed in synchronism with the system at $r=R$. Hence the difference between equation (14) and equation (62) is a real and necessary one.

3.7. *Geodesic equations derived from collated proper measurements*

In this section we consider events in the rotating system as described by a single rotating observer who makes 'proper' measurements within his own locality as he moves from point to point within the rotating system or, alternatively, events as described by an infinity of rotating observers at all points upon the rotating system, each of whom makes proper measurements within his own locality. The measurements are all collated at a common data centre. We obtain these measurements by letting R vary, i.e. we let $R=r$. We have already obtained the expression for the proper measurement of particle velocity u' upon the rotating system, namely equation (32) which, for a null geodesic, i.e. $u=c$, gives $u'=c$ thus satisfying a basic postulate of relativity. Let us now examine the proper measurements of geodesics, null-geodesics and spatial geodesics within the rotating system, as made by observers in synchronous rotation with the system.

Geodesics

The proper measurement of $\mathrm{d}\theta'/\mathrm{d}r$ for a geodesic may be obtained from equation (48) by setting $R=r$:

$$\frac{\mathrm{d}\theta'}{\mathrm{d}r}=\pm\frac{(au\pm\omega r^2)}{ur(r^2-a^2)^{1/2}(1-r^2\omega^2/c^2)^{1/2}} \tag{63}$$

which, since u is a constant, may be integrated to give

$$\theta' = \pm\left\{\sin^{-1}\left[\frac{c(r^2-a^2)^{1/2}}{r(c^2-a^2\omega^2)^{1/2}}\right] \pm \frac{c}{u}\sin^{-1}\left[\frac{\omega(r^2-a^2)^{1/2}}{(c^2-a^2\omega^2)^{1/2}}\right]\right\}. \tag{64}$$

From equation (50) it can be seen that $d\sigma'$ is independent of R, hence σ' may be obtained directly from equation (51) as

$$\sigma' = \frac{(c^2 \pm a\omega u)}{\omega u}E(x, k) - \frac{(c^2-u^2)(r^2-a^2)^{1/2}(c^2-\omega^2r^2)^{1/2}}{u[(c^2-a\omega u)^2-(c^2-u^2)(c^2-\omega^2r^2)]^{1/2}} \tag{65}$$

in which $E(x, k)$, x and k have the same meanings as previously.

A proper measurement of the time taken for a particle to travel a distance $d\sigma'$ is given by equation (30), namely

$$dT' = dt' - R^2\omega\, d\theta'/c^2 \tag{66}$$

into which we can substitute for dt' from equation (54) (after setting $R = r$) and for $d\theta'$ from equation (63), thus obtaining

$$dT' = \pm\frac{cr(1\pm\omega au/c^2)\,dr}{u(r^2-a^2)^{1/2}(c^2-\omega^2r^2)^{1/2}} \tag{67}$$

which, since u is constant, integrates to give

$$T' = \frac{c}{\omega u}\left(1\pm\frac{\omega au}{c^2}\right)\sin^{-1}\left\{\frac{(\omega/c)(r^2-a^2)^{1/2}}{(1-a^2\omega^2/c^2)^{1/2}}\right\}. \tag{68}$$

Null geodesics

The proper measurement of $d\theta'/dr$ within the system S' may be obtained from equation (63) by setting $u = c$. Hence

$$\theta' = \pm\cos^{-1}\left\{\frac{ac\pm\omega r^2}{r(c\pm a\omega)}\right\} \tag{69}$$

if $\theta' = 0$ when $r = a$. Setting $u = c$ in equations (65) and (68) gives

$$\sigma' = \frac{1}{\omega}(c \bullet a\omega)\sin^{-1}\left(\frac{(\omega/c)(r^2-a^2)^{1/2}}{(1-\omega^2a^2/c^2)^{1/2}}\right) \tag{70}$$

and

$$T' = \frac{(c\pm a\omega)}{\omega c}\sin^{-1}\left(\frac{\omega}{c}\,\frac{(r^2-a^2)^{1/2}}{(1-\omega^2a^2/c^2)^{1/2}}\right). \tag{71}$$

Equations (69) and (70) describe the paths of circular arcs and have been discussed by Ashworth and Davies (1977).

Spatial geodesics

The equations for a spatial geodesic are obtained by setting $u = c^2/a\omega$ in equations (64), (65) and (68) giving

$$\theta' = \pm\left[\sin^{-1}\left(\frac{c(r^2-a^2)^{1/2}}{r(c^2-a^2\omega^2)^{1/2}}\right) - \frac{a\omega}{c}\sin^{-1}\left(\frac{\omega(r^2-a^2)^{1/2}}{(c^2-a^2\omega^2)^{1/2}}\right)\right], \tag{72}$$

$$\sigma' = (1 - a^2\omega^2/c^2)^{1/2}(r^2 - a^2)^{1/2} \tag{73}$$

and

$$T' = 0. \tag{74}$$

Equation (74) is a direct consequence of the fact that $u' = \infty$ for a spatial geodesic.

The above geodesic equations have been written in terms of r, a and ω rather than r', a' and ω' in order to keep the equations as simple as possible. Equations (70) and (73) were first derived by Ashworth and Jennison (1976).

3.8. Aberration angles

If ϕ' is the angle between the positive direction of motion of the light ray (when passing through the locality of an observer at $r = R$) and the direction of the velocity vector of the observer at $r = R$ (according to an observer who is stationary in the laboratory system S) then from equation (25)

$$\tan\phi' = \frac{\mathrm{d}r(1 - r^2\omega^2/c^2)^{1/2}}{r(1 - R^2\omega^2/c^2)^{1/2}\,\mathrm{d}\theta'} \tag{75}$$

which, when combined with equation (63) (after setting $u = c$), gives

$$\tan\phi' = \pm\frac{c(R^2 - a^2)^{1/2}(1 - R^2\omega^2/c^2)^{1/2}}{ac \pm \omega R^2} \tag{76}$$

after setting $r = R$. equation (76) is the 'aberration' equation of special relativity and is an experimentally verifiable expression.

3.9. Electric and magnetic fields in the rotating system

Using the transformations of equation (20) it can readily be shown (see e.g. Atwater 1974) that an electric field in the inertial frame S whose components are E_r, E_θ, E_z and a magnetic field in the inertial frame S with components of magnetic induction given by B_r, B_θ, B_z will transform into an electric field with components E'_r, E'_θ, E'_z in S' and a magnetic field with components of magnetic induction B'_r, B'_θ, B'_z where

$$\begin{aligned}
E_r &= (E'_r - \omega' r' B'_z)(1 + R'^2\omega'^2/c^2)^{-1} & E'_r &= (E_r + \omega r B_z)(1 - R^2\omega^2/c^2)^{-1} \\
E_\theta &= E'_\theta(1 + R'^2\omega'^2/c^2)^{-1/2} & E'_\theta &= E_\theta(1 - R^2\omega^2/c^2)^{-1/2} \\
E_z &= (E'_z + \omega' r' B'_r)(1 + R'^2\omega'^2/c^2)^{-1/2} & E'_z &= (E_z - \omega r B_r)(1 - R^2\omega^2/c^2)^{-1/2}
\end{aligned} \tag{77}$$

and

$$\begin{aligned}
B_r &= B'_r & B'_r &= B_r \\
B_\theta &= B'_\theta(1 + R'^2\omega'^2/c^2)^{-1/2} & B'_\theta &= B_\theta(1 - R^2\omega^2/c^2)^{-1/2} \\
B_z &= B'_z(1 + R'^2\omega'^2/c^2)^{-1/2} & B'_z &= B_z(1 - R^2\omega^2/c^2)^{-1/2}.
\end{aligned} \tag{78}$$

In addition current density and charge density written in the form of a four vector transform according to equation (20).

3.10. Equation of motion in the rotating system for a charged test particle in an electromagnetic field

The equations of motion for a charged test particle in an electromagnetic field can be found by using equation (47) together with the metric of equation (22) (Atwater 1974) giving

$$\begin{aligned}
&\ddot{r}' - r'\gamma^2[\dot{\theta}' + \omega'\dot{t}']^2 = (q\gamma^2/mc)(\dot{t}'E'_r + \dot{\theta}'r'B'_z - \dot{z}'B'_\theta)\\
&\ddot{\theta}' + (2\dot{r}'/r')(\dot{\theta}' + \omega'\dot{t}') = (q\gamma^4/mcr')(\dot{t}'E'_\theta - \dot{r}'B'_z + \dot{z}'B'_r) - \omega' f''\\
&\ddot{z}' = (q/mc)(\dot{t}'E'_z + \dot{r}'B'_\theta - \dot{\theta}'r'B'_r)\\
&\ddot{t}' = (q\gamma^2/mc^3)[r'\omega'(\dot{t}'E'_\theta - \dot{r}'B'_z + \dot{z}'B'_r) + \dot{r}'E'_r + \dot{\theta}'r'E'_\theta + \dot{z}'E'_z] = f''
\end{aligned} \tag{79}$$

in which

$$\gamma^2 = (1 + R'^2\omega'^2/c^2)^{-1}, \tag{80}$$

m is the rest mass of the test particle, q is the charge on the test particle and denotes differentiation with respect to s. It is readily shown that

$$\mathrm{d}s = c\ \mathrm{d}t'\left(\frac{[c^2 + \omega'^2(R'^2 - r'^2) - r'^2\omega'\ \mathrm{d}\theta'/\mathrm{d}t']^2}{c^2[c^2 + \omega'^2(R'^2 - r'^2)]} - \frac{u'^2_{\text{app}}}{c^2}\right)^{1/2} \tag{81}$$

where, as previously (see equation (38))

$$u'_{\text{app}} = \mathrm{d}\sigma'/\mathrm{d}t' \tag{82}$$

is the apparent or coordinate velocity. In the proper frame of the observer, i.e. when $r' = R'$, we find that

$$\mathrm{d}s = c\ \mathrm{d}T'(1 - u'^2/c^2)^{1/2} \tag{83}$$

where, as previously (see Eq. 31))

$$u' = \mathrm{d}\sigma'/\mathrm{d}T' \tag{84}$$

is the measured velocity at $r' = R'$.

3.11. The cyclotron equation

Let us consider a charged particle which is stationary with respect to the rotating coordinate system, i.e. if proper measurements are made then

$$u' = \frac{\mathrm{d}u'}{\mathrm{d}T'} = \frac{\mathrm{d}^2r'}{\mathrm{d}T'^2} = \frac{\mathrm{d}r'}{\mathrm{d}T'} = \frac{\mathrm{d}^2\theta'}{\mathrm{d}T'^2} = \frac{\mathrm{d}\theta'}{\mathrm{d}T'} = \frac{\mathrm{d}^2z'}{\mathrm{d}T'^2} = \frac{\mathrm{d}z'}{\mathrm{d}T'} = 0. \tag{85}$$

Equations (79) now simplify to give

$$-mR'\omega'^2 = qE'_R \qquad E'_\theta = 0 \qquad E'_z = 0 \tag{86}$$

which, by equations (17), (20), (77) and (78) give

$$\frac{-mR\omega^2}{(1 - R^2\omega^2/c^2)^{1/2}} = q(E_R + \omega RB_z) \tag{87}$$

$$E_\theta = 0, \qquad E_z = \omega RB_R \tag{88), (89}$$

For an electron $q = -e$ and, if $E_R = 0$, equation (87) gives

$$\frac{mR\omega^2}{(1-R^2\omega^2/c^2)^{1/2}} = B_z eR\omega \tag{90}$$

which is the relativistic cyclotron equation and has been verified experimentally.

4. Conclusions

We have considered, in detail, the problem of relating measurements made by an observer at rest in an inertial laboratory frame to measurements made by an observer who is in synchronous rotation with a rotating system. We have assumed that the rotating system does not contain any matter other than test particles.

Starting from first principles we have derived the appropriate coordinate transformations and have then applied these to the Minkowski metric to produce a general metric for a rotating system as defined above. It is important to realise that the metric is derived for an observer who is stationary with respect to the rotating system at inertial radius R rather than the more usual metric applied to rotating systems which describes the space–time as seen by an observer at the centre of rotation who is rotating with the system. These two metrics are obviously identical for $R = 0$. Using our metric we have examined various fundamental properties of the space. These include the various geodesics for the system and the equations of motion for charged and uncharged test particles and photons. A point of particular interest in connection with the geodesic analyses is the comparison of the geodesics obtained directly from the metric and those obtained by means of 'collated proper measurements'. 'Collated proper measurements' are obtained when we allow the parameter R to vary, that is we let $R = r$. Hence the various equations derived by this technique correspond to a description of events made by an observer who actually moves through the rotating system and who makes measurements at various points, each measurement being made while he is at rest with respect to the rotating system. These measurements are then collated and the corresponding geodesic paths are plotted out.

Throughout the paper we indicate where local measurements using the metric we have derived produce results identical with those obtained by the instantaneous Lorentz frame approach to the analysis of a rotating system. A particular example is the equation for the aberration angle of a light ray seen by a rotating observer.

We have also considered the description of electromagnetic fields in the rotating system and we have derived the equations of motion for a charged test particle in the rotating system. We have derived the cyclotron equation and we consider this result to be strong support for our metric. The equation is derived from the equations of motion for a charged test particle within the rotating system rather than the more usual technique of deriving the cyclotron equation from the equation of motion of a particle within the laboratory frame.

In conclusion let us state once again that the results of the paper are a function of the way measurements are made in rotating frames. However, having said this, we feel that the results are very important in the interpretation of these measurements. The paper relates, in a precise manner, the 'metric' and the 'instantaneous Lorentz frame' approaches to the problems of rotating systems. It is now possible to analyse problems in rotating systems through either of the two techniques and to know that both techniques give compatible results providing that one is careful to define exactly what one is measuring, the method of making the measurements and the way in which they are interpreted. However, the 'metric' approach is far more versatile in its applications without leading to any increase in complexity.

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